DIFFERENTIAL EQUATIONS

INITIAL VALUE PROBLEM: let $D \subset \mathbb{R} \times \mathbb{R}^d$ open and $f: D \to \mathbb{R}^d$,

ODE $\to$ $\dot{x} = f(t,x)$ w/ $x(t_0) = x_0$ for $(t_0, x_0) \in D$ is IVP,

and $\lambda : I \to \mathbb{R}^d$ sol^n s.t. $\lambda(t_0) = x_0$.

IVP w/ no sol^n: $\dot{x} = \begin{cases} 1 & : x < 0 \\ -1 & : x \geq 0 \end{cases}$ $\quad x(0) = 0$

$\leftarrow$ discontinuous RHS.

IVP w/ multiple sol^{n}s: $\dot{x} = \sqrt{|x|}$, $x(0) = 0$

$\lambda_b(t) = \begin{cases} 0 & : t \leq b \\ \frac{1}{4}(t-b)^2 & : t > b \end{cases}$ $\to$ for any $b \geq 0$.

TRANSLATIONAL INVARIANCE: $\lambda : I \to \mathbb{R}^d$ sol^n to autonomous DE

$\dot{x} = f(x)$, then $\forall \tau \in \mathbb{R}$ $\mu(t) = \lambda(t+\tau)$ also a sol^n

for $t \in \tilde{I} = \{t \in \mathbb{R} : t + \tau \in I\}$.

$\lambda : I \to \mathbb{R}^d$ solves IVP $\iff$ $\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\, ds$ $\quad (\forall t \in I)$

$\hookrightarrow$ so $\dot{\lambda}(t) = f(t, \lambda(t))$

PICARD ITERATES: $\lambda_0(t) = x_0$ $\forall t \in J$ where J interval containing t_0.

$$\Rightarrow \lambda_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\, ds \quad \forall t \in J.$$

LIPSCHITZ CONTINUITY: $f : X \to Y$ (normed vector spaces)

then, $\|f(x) - f(y)\| \leq K \cdot \|x - y\|$ $\quad \forall x, y \in X$.

by MVT $|f(x) - f(y)| = |f'(\xi)| \cdot |x - y| \leq M|x - y|$

so f' bounded $\Rightarrow$ f Lipschitz.

(also applies in higher dimensions where $\|f'(\xi)\|$ is matrix instead)

OPERATOR NORM: $\|A\| = \sup \dfrac{\|Ax\|}{\|x\|}$ = sqrt of largest evalue of A^*A.

$\exists K_1, K_2$ s.t. $K_1 \|A\|_{max} \leq \|A\| \leq K_2 \|A\|_{max}$ $\quad$ (where $\|A\|_{max} = \max(a_{ij})$)

PICARD-LINDELÖF THM (global): consider an ODE, $\dot{x} = f(t,x)$, s.t. $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ continuous + globally Lipschitz,

$$\|f(t,x) - f(t,y)\| \leq K \cdot \|x-y\| \quad \forall t \in \mathbb{R}, \; x, y \in \mathbb{R}^d.$$

Define $h := \frac{1}{2K}$ then every IVP with $x(t_0) = x_0$ has a unique sol^n $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$.

↳ pf by Banach's fixed point thm $P : C^0([t_0-h, t_0+h], \mathbb{R}^d) \to C^0([t_0-h, t_0+h], \mathbb{R}^d)$ where P contraction $P(u(t)) = x_0 + \int_{t_0}^{t} f(s, u(s))\, ds$ and fixed point is soln to ODE. Note these are Picard iterates.

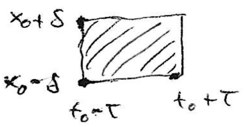
PICARD-LINDELÖF THM (local): Let $D \subset \mathbb{R} \times \mathbb{R}^d$ and $f : D \to \mathbb{R}^d$ continuous + locally Lipschitz w.r.t. x. Consider IVP,

$$\dot{x} = f(t,x), \quad x_0 = x(t_0).$$

for $\tau, \delta > 0$, $W^{\tau,\delta}(t_0, x_0) = [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)} \subset D$.

Suppose $\exists K, M > 0$ s.t. $\|f(t,x) - f(t,y)\| \leq K\|x-y\|$

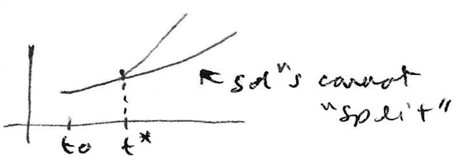
$\|f(t,x)\| \leq M$ $\quad \forall$ in $W^{\tau,\delta}(t_0,x_0)$

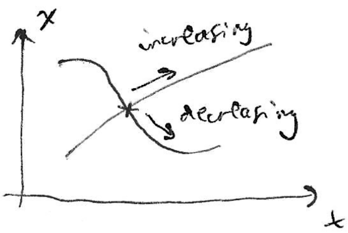


then IVP has exactly one sol^n on $[t_0 - h, t_0 + h]$ with

$$h = \min\left\{\tau, \frac{1}{2K}, \frac{\delta}{M}\right\}$$

SOL^{N}S CANNOT CROSS:





never possible due to uniqueness in Picard-Lindelöf thm.

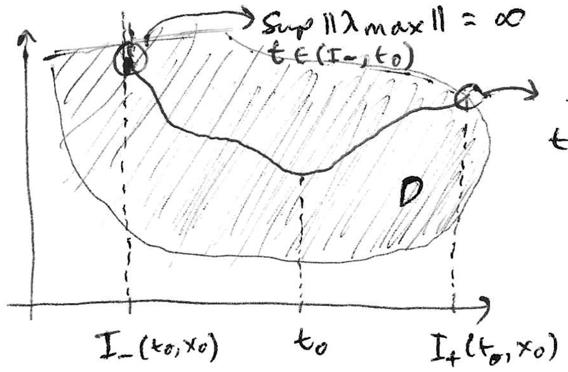
MAXIMUM EXISTENCE INTERVAL:

$$I_+(t_0, x_0) = \sup\{t_+ \geq t_0 : \exists \text{ sol}^n \text{ to IVP on } [t_0, t_+]\}.$$

$$I_-(t_0, x_0) = \inf\{t_- \leq t_0 : \exists \text{ sol}^n \text{ to IVP on } [t_-, t_0]\}.$$

↳ then $I_{max}(t_0, x_0) = (I_-(t_0,x_0), I_+(t_0,x_0))$.

• maximal sol^n is unique sol^n w/ biggest possible domain.



← sol^n has to be unbounded or must converge into the boundary of the domain.

N/B: maximal sol^n → $\lambda_{max} : I_{max}(t_0, x_0) \to \mathbb{R}^d$.

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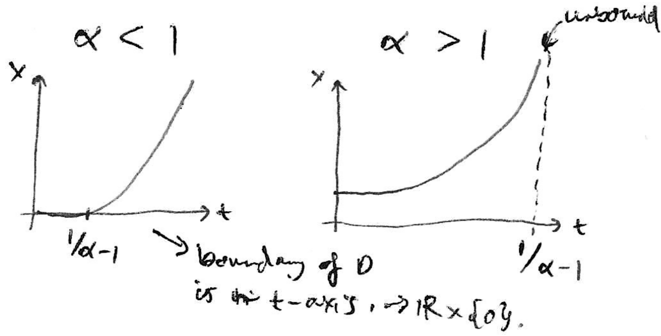
e.g. for $\alpha > 0$, $\dot{x} = x^\alpha$ (autonomous) [$D = \mathbb{R} \times \mathbb{R}^+$ since time is $\mathbb{R}$ but x restricted to $\mathbb{R}^+$]

if $\alpha = 1$, $\lambda_{max} = e^t$. ($x(0) = 1$).

for $\alpha \neq 1$, $\lambda_{max} = (1 + (1-\alpha)t)^{\frac{1}{1-\alpha}}$.

↳ need $(1 + (1-\alpha)t) > 0$. $\forall t$.

$$I_{max} = \begin{cases} (\frac{1}{\alpha - 1}, \infty), & \alpha \in (0,1) \\ (-\infty, \infty), & \alpha = 1 \\ (-\infty, \frac{1}{\alpha-1}), & \alpha \in (1, \infty) \end{cases}$$



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general sol^n → $\lambda_{max}(t, t_0, x_0)$

↳ so $\lambda(t_0, t_0, x_0) = x_0$ and $\lambda(t, s, \lambda(s, t_0, x_0)) = \lambda(t, t_0, x_0)$ $\forall t \in I_{max}(t_0, x_0)$

FLOW of AUTONOMOUS DE: define $J_{max}(x_0) = I_{max}(0, x_0)$

$\dot{x} = f(x)$ and let $\varphi(t, x_0) = \lambda(t, 0, x_0)$ $\forall t \in J_{max}$ flow.

PROPERTIES: $\varphi(0, x) = x$, $\varphi(t, \varphi(s, x)) = \varphi(t+s, x)$

ORBIT: $O(x) = \{\varphi(t, x) \in D : t \in J_{max}(x)\}$.

O^+ is same for +ve time → $t \in J_{max}(x) \cap \mathbb{R}_0^+$.

O^- has $t \in J_{max}(x) \cap \mathbb{R}_0^-$.

3 different types of orbits : • SINGLETON ($\Rightarrow f(x) = 0$)

here, $O(x) = x$ and $J_{max}(x) = \mathbb{R}$

↳ x is equilibrium point in this case.

• $O(x)$ PERIODIC ORBIT → closed curve ($\exists t > 0$ s.t. $\varphi(t,x) = x$)

$J_{max} = \mathbb{R}$.

• $O(x)$ not closed curve → $t \mapsto \varphi(t,x)$ is injective on J_{max}.

N/B: in 1-D case all sol^{n}s MONOTONE ⇒ doesn't exist periodic orbits.

LINEAR SYSTEM : $\dot{x} = Ax$ ($\dot{x} = f(x) \to \dot{x} = f'(x^*) \cdot x$).

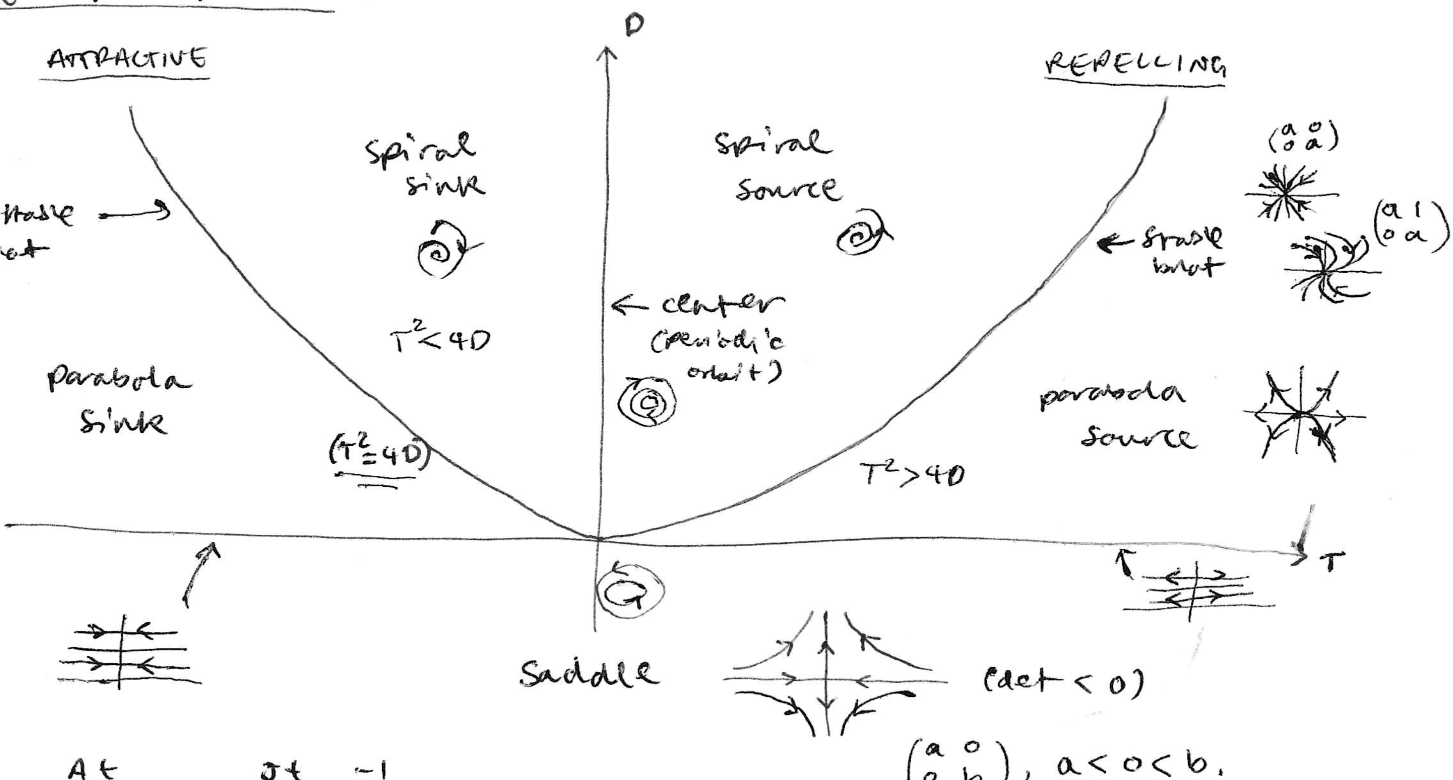
notice → $\|Ax - Ay\| \leq \|A\|\|x-y\| \leq K\|x-y\|$ → Lipschitz.

sol^n is $e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$, so $\varphi(t,x) = e^{At} \cdot x$.

MATRIX EXPONENTIAL : if $BC = CB \Rightarrow e^{B+C} = e^B \cdot e^C$.

if $C = T^{-1}BT \Rightarrow e^C = T^{-1}e^B T$.

PHASE PORTRAITS :



for $e^{At} = Te^{Jt}T^{-1}$ → applying linear transformation T to original phase portrait w/ J.

LYAPUNOV EXPONENT : $\sigma(\lambda) = \lim_{t \to \infty} \frac{\ln\|\lambda(t)\|}{t}$ → exponential growth rate.

given evalue $a+ib$, a characterises exponential growth.
b characterises rotation. $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ look at this b.

- if $\sigma(\lambda)$ = LYAPUNOV constant >0 exponential increase.
  otherwise exponential decay.
- similarly $b<0$ anticlockwise + vice-versa.

VARIATION of CONSTANTS : $\dot{x} = Ax + g(t)$

$$\Rightarrow \lambda(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)} g(s)\, ds.$$

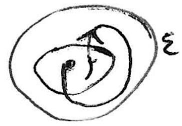
NOTIONS of STABILITY (non-linear systems) : x^* equilibrium

- STABLE if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.
  $\| \varphi(t,x) - x^* \| < \varepsilon \quad \forall x \in B_\delta(x^*), t \geq 0$

- UNSTABLE if not stable,

- ATTRACTIVE if $\exists \delta > 0$ s.t.
  $\lim_{t\to\infty} \varphi(t,x) = x^* \quad \forall x \in B_\delta(x^*)$

- ASYMPTOTICALLY STABLE if stable + attractive.



- REPULSIVE if attractive backward in time.
  ↳ i.e. $\lim_{t\to-\infty} \varphi(t,x) = x^* \quad \forall x \in B_\delta(x^*)$.
- EXPONENTIALLY STABLE if $\exists \delta > 0, K \geq 1, \gamma < 0$ s.t.
  $\| \varphi(t,x) - x^* \| \leq K e^{\gamma t} \| x - x^* \| \quad \forall x \in B_\delta(x^*), t \geq 0.$

N/B : exponentially stable $\Rightarrow$ asymptotically stable.
(but no relation between stable + attractive).

HOMOCLINIC orbit : $\lim_{t\to\infty} \varphi(t,x) = x^*, \lim_{t\to-\infty} \varphi(t,x) = x^*$

HETEROCLINIC orbit : $\lim_{t\to\infty} \varphi(t,x) = x_1^*, \lim_{t\to-\infty} \varphi(t,x) = x_2^*$

STABILITY of TRIVIAL EQUILIBRIUM: $\dot{x} = Ax$, $x^* = 0$.

- exponentially stable iff $\mathrm{Re}(\rho) < 0$ $\forall$ evalues ρ.
- stable iff $\mathrm{Re}(\rho) \leq 0$ with $\mathrm{Re}(\rho) = 0$ if alg mult = geo mult of ρ.

HYPERBOLICITY: matrix A hyperbolic if all evalues of A have NON-ZERO REAL part ($\mathrm{Re}(\lambda) \neq 0$).

N/B: motivation for this → matrices w/ purely imaginary evalues, the nonlinearly perturbed system behaves much differently than linear one.

$\boxed{\text{LINEARISED STABILITY: } x^* \text{ exponentially stable iff } \mathrm{Re}(\lambda) < 0 \ \forall \lambda}$

STABLE/UNSTABLE SET: $W^s(x^*) = \{x \in D : \lim_{t\to\infty} \varphi(t,x) = x^*\}$.
$W^u(x^*) = \{x \in D : \lim_{t\to-\infty} \varphi(t,x) = x^*\}$.

(all points in domain which converge to x^* forward in time).
↳ if x^* attractive → $W^s(x^*)$ known as "domain of attraction".

- x^* attractive ⇒ $\exists \delta > 0$ s.t. $\forall x \in B_\delta(x^*) \subset W^s(x^*)$.

INVARIANCE: positively invariant $M \subset D$ if $\forall x \in M$, $O^+(x) \subset M$.
↳ if a point starts in M, it stays in M forward in time.

- negatively invariant if $\forall x \in M$, $O^-(x) \subset M$.
- invariant if $\forall x \in M$, $O(x) \subset M$. → e.g. homoclinic/heteroclinic orbits.

LIMIT SETS: $x_\omega \in D$ is ω-limit point if $\exists$ sequence t_n s.t

$$\lim_{n\to\infty} t_n = \infty \quad \text{and} \quad x_\omega = \lim_{n\to\infty} \varphi(t_n, x).$$

(evolve ~~sequence~~ flow along sequence of times t_n, converge to x_ω).

$\omega(x)$ = set of all $x_\omega \in D$.

$\alpha(x)$ same just with sequence $t_n \to -\infty$ instead.

alternatively, $\omega(x) = \bigcap_{t\geq 0} \overline{O^+(\varphi(t,x))}$.

• $\omega(x)$ and $\alpha(x)$ are INVARIANT,

(also if $O^+(x)$ bounded and $\overline{O^+(x)} \subset D$ then $\omega(x)$ compact + non-empty)

LYAPUNOV FUNCTIONS: applicable to non-hyperbolic equilibria + gives global information.

↳ $V: D \to \mathbb{R}$ real-valued scalar funct^{n}s ("energy").

ORBITAL DERIVATIVE: $\dot{V}(x) = V'(x) \cdot f(x)$

↖ Jacobian

$V: D \to \mathbb{R}$ Lyapunov funct^n if $\underline{\dot{V}(x) \leq 0}$ $\forall x \in D$.

N/B: $V(\varphi(t,x)) \leq V(x)$ → so Lyapunov funct^n decreases along sol^{n}s.

SUBLEVEL SETS +vely INVARIANT: $S_c = \{x \in D : V(x) \leq c\}$

"if I start in S_c, I must stay in S_c forward in time".

LYAPUNOV DIRECT METHOD for STABILITY: if x^* local minimum of V, i.e. $V(x^*) = 0$ and $V(x) > 0$, then $\underline{x^* \text{ STABLE}}$.

LA SALLE'S INVARIANCE PRINCIPLE:

$$\omega(x) \subset \{y \in D : \dot{V}(y) = 0\}$$

→ $\omega(x)$ is subset of zero-level set of orbital derivative.

coroll: $\omega(x) \subset$ union of invariant subsets of $\{y \in D : \dot{V}(y) = 0\}$.

(apply this for asymptotic stability).

LYAPUNOV DIRECT METHOD for ASYMPTOTIC STABILITY:

$$V(x^*) = 0 \quad \text{and} \quad V(x) > 0 \quad (\forall x \in D \setminus \{x^*\})$$

$$\dot{V}(x^*) = 0 \quad \text{and} \quad \dot{V}(x) < 0 \quad (\forall x \in D \setminus \{x^*\})$$

⇒ x^* ASYMPTOTICALLY STABLE.

coroll: Sublevel sets of Lyapunov funct^{n}s are part of domain of attraction.

SUBLEVEL SETS are SUBSETS of DOMAIN of ATTRACTION!

$S_c = \{x \in D : V(x) \leq c\}$, then $S_c \subset W^s(x^*)$ if

$S_c \subset D$ is compact.

POINCARE - BENDIXSON THM: consider $\dot{x} = f(x)$ and

for some $x \in D$, $O^+(x) \subset K$ compact subset of D, containing

finitely many equilibria. Then one of following hold:

- $\omega(x)$ is a singleton of one equilibrium.
- $\omega(x)$ is a periodic orbit
- $\omega(x)$ consists of equilibria + non-closed orbits
  (so either homoclinic or heteroclinic orbits).

↳ note → analogous statement for $\alpha(x)$ in negative half orbit.

- P-B thm shows only very regular behaviour in 2D case.

COROLLARY of P-B (existence of PERIODIC ORBIT):

if $O^+(x) \subset K$ compact subset of D and K does NOT contain an equilibrium, then $\omega(x)$ is PERIODIC ORBIT.